

A conformal scalar dyon black hole solution

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Abstract

An exact solution of Einstein - Maxwell - conformal scalar field equations is given, which is a black hole solution and has three parameters: scalar charge, electric charge, and magnetic charge. Switching off the magnetic charge parameter yields the solution given by Bekenstein. In addition the energy of the conformal scalar dyon black hole is obtained.

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It is known that Maxwell's equations are conformally invariant in four dimensions, whereas the ordinary massless scalar equation is conformally invariant in two dimensions. However, there are scalar equations, which are conformally invariant in any arbitrary finite dimension (dimension greater than one) [1]. For instance, in four dimensions one has the following conformally invariant scalar equation :

$$\Psi_{;i}{}^{;i} - \frac{1}{6}R\Psi = 0 \quad (1)$$

Ψ is the conformal scalar field and R is the scalar curvature. Comma and semicolon denote, respectively, the partial and covariant derivatives. We use the geometrized units ($G = 1, c = 1$) and follow the convention that the Latin indices take values 0 to 3 and Greek indices take values 1 to 3. x^0 is the time coordinate. The energy-momentum tensor for the conformal scalar field given by Eq. (1) differs from that of the ordinary massless scalar field and it has several interesting properties [2 – 4]. Inspired by this, Bekenstein [4] obtained a static and spherically symmetric exact solution of Einstein-Maxwell-conformal scalar field equations which is characterized by two parameters: scalar charge and electric charge. The solution obtained by Bekenstein is asymptotically flat and has an event horizon, but the scalar field diverges on the event horizon. Therefore, he suggested that the solution cannot be interpreted as a black hole solution. However, his subsequent analysis [5] revealed that, the infinity in the conformal scalar field is not associated with an infinitely high potential barrier, no test particle trajectories terminate at horizon at finite proper time, and tidal accelerations remain bounded at the horizon. He concluded that the horizon is physically regular and therefore the solution obtained by him is a black hole solution. It was conjectured that the stationary black holes could be parametrized only by mass, angular momentum, electric and magnetic charges and this was paraphrased by Wheeler as "black holes have no hair". Having shown that the solution obtained by him is a black hole solution, Bekenstein stressed that the scalar charge should also be included as a black hole parameter.

Although the existence of magnetic monopoles is not yet confirmed, it has drawn attention of many theoreticians (see in reference 6). As Bekenstein's solution is not enriched by the

magnetic charge parameter, it is of interest to obtain a conformal scalar dyon (CSD) black hole solution (of Einstein-Maxwell-conformal scalar field equations) which is characterized by scalar, electric and magnetic charges.

The Einstein-Maxwell-conformal field equations [4] are

$$R_{ij} - \frac{1}{2}g_{ij}R = 8\pi\Sigma_{ij} \quad (2)$$

where

$$\Sigma_{ij} = (\Theta_{ij} + E_{ij}) (1 - \zeta^2 \Psi^2)^{-1} \quad (3)$$

$$\Theta_{ij} = S_{ij} - \frac{1}{6}(\Psi^2)_{,i;j} + \frac{1}{6}(\Psi^2)_{,l;m} g^{lm} g_{ij} \quad (4)$$

$$S_{ij} = \Psi_{,i} \Psi_{,j} - \frac{1}{2}g_{ij} g^{kl} \Psi_{,k} \Psi_{,l} \quad (5)$$

$$E_{ij} = \frac{1}{4\pi} \left[F_{ik} F_{jl} - \frac{1}{4}g_{ij} g^{mn} F_{mk} F_{nl} \right] g^{kl} \quad (6)$$

(S_{ij} and E_{ij} , respectively, are the energy-momentum tensors of the ordinary scalar field and electromagnetic field)

$$\zeta = \sqrt{\frac{4\pi}{3}} \quad (7)$$

$$(\sqrt{-g} F^{ij})_{,j} = 0 \quad (8)$$

$$(\sqrt{-g} {}^*F^{ij})_{,j} = 0 \quad (9)$$

${}^*F^{ij}$, the dual of the electromagnetic field tensor F^{ij} , is

$${}^*F^{ij} = \frac{1}{2\sqrt{-g}} \varepsilon^{ijkl} F_{kl} \quad (10)$$

(ε^{ijkl} is the Levi-Civita tensor density). Recall that the conformal scalar equation is given by Eq. (1).

An exact solution of the above equations, characterized by the scalar charge, electric charge, and magnetic charge is given by the line element,

$$ds^2 = \left(1 - \frac{Q_{CSD}}{r}\right)^2 dt^2 - \left(1 - \frac{Q_{CSD}}{r}\right)^{-2} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2) \quad (11)$$

the conformal scalar field,

$$\Psi = \sqrt{\frac{3}{4\pi}} \left(\frac{q_s}{r - Q_{CSD}} \right) \quad (12)$$

where

$$Q_{CSD} = \sqrt{q_s^2 + q_e^2 + q_m^2} \quad (13)$$

and the non-vanishing components of the electromagnetic field tensor,

$$F_{rt} = \frac{q_e}{r^2} \quad (14)$$

$$F_{\theta\phi} = q_m \sin\theta \quad (15)$$

where q_s , q_e , and q_m , respectively, denote the scalar charge, electric charge, and magnetic charge. For convenience we call Q_{CSD} *conformal scalar dyon charge*, which is the square root of the sum of the squares of the scalar, electric, and magnetic charges. The solution given by us has an event horizon at $r = Q_{CSD}$.

For the CSD black hole solution, given by (11) – (15), the non-vanishing components of Θ_k^i and E_k^i are

$$\Theta_0^0 = \Theta_1^1 = -\Theta_2^2 = -\Theta_3^3 = \frac{q_s^2 (r - 2Q_{CSD})}{8\pi r^3 (r - Q_{CSD})^2} \quad (16)$$

$$E_0^0 = E_1^1 = -E_2^2 = -E_3^3 = \frac{q_e^2 + q_m^2}{8\pi r^4} \quad (17)$$

and the non-vanishing components of the Einstein tensor $G_j^i \equiv R_j^i - \frac{1}{2}g_j^i R$ and Σ_j^i are

$$G_0^0 = G_1^1 = -G_2^2 = -G_3^3 = 8\pi\Sigma_0^0 = 8\pi\Sigma_1^1 = -8\pi\Sigma_2^2 = -8\pi\Sigma_3^3 = \frac{Q_{CSD}^2}{r^4} \quad (18)$$

It is of interest to obtain the gravitational energy of the CSD black hole. There are many prescriptions for obtaining energy and momentum in curved spacetimes [7]. Among them Weinberg's prescription is one of the most handy. The expression for energy in Weinberg's prescription is [8]

$$E = \frac{1}{16\pi} \int \int \left(\frac{\partial h_\alpha^\alpha}{\partial x_\beta} - \frac{\partial h^{\alpha\beta}}{\partial x^\alpha} \right) n_\beta r^2 \sin\theta d\theta d\phi \quad (19)$$

where $n_1 = x/r$, $n_2 = y/r$, $n_3 = z/r$, $r^2 = x^2 + y^2 + z^2$ and

$$h_{ij} = g_{ij} - \eta_{ij} \quad (20)$$

η_{ij} is the Minkowski metric and indices on h_{ij} or $\frac{\partial}{\partial x^i}$ are raised or lowered with help of η'_{ij} . It is known that Weinberg's prescription for obtaining energy of a general relativistic system gives correct result if calculations are carried out in coordinates in which the metric g_{ij} approaches the Minkowski metric η_{ij} at great distances from the system under study. Therefore, one writes the metric (11) in Kerr-Schild Cartesian coordinates (T, x, y, z) [9].

$$ds^2 = dT^2 - dx^2 - dy^2 - dz^2 - \frac{2Q_{CSD} \left(1 - \frac{Q_{CSD}}{2r}\right)}{r} \left[dT + \frac{xdx + ydy + zdz}{r} \right]^2 \quad (21)$$

The coordinates t, r, θ, ϕ in (11) and T, x, y, z in (21) are related through

$$T = t - r + \int \left(1 - \frac{Q_{CSD}}{r}\right)^{-2} dr$$

$$x = r \sin\theta \cos\phi$$

$$\begin{aligned}
y &= r \sin\theta \sin\phi \\
z &= r \cos\theta
\end{aligned}
\tag{22}$$

With (20) and (21), one has

$$h_{\alpha\beta} = \frac{-2Q_{CSD} \left(1 - \frac{Q_{CSD}}{2r}\right)}{r^3} x^\alpha x^\beta \tag{23}$$

Using (23) in (19), a straightforward calculation yields

$$E = Q_{CSD} \left(1 - \frac{Q_{CSD}}{2r}\right) \tag{24}$$

Therefore, the total gravitational energy of a CSD black hole (r approaching infinity in Eq. (24)) is given by its CSD charge.

We have given an exact solution of CSD black hole which has three parameters: scalar, electric, and magnetic charges. This solution is the magnetic charge generalization of the Bekenstein solution. It is obvious that the metric of the CSD black hole is the same as the well known Reissner-Nordström (R-N) metric when mass and charge parameters both are set to the CSD charge. Like the R-N geometry this has a curvature singularity at $r = 0$. However, it differs from the R-N metric as it has only one horizon ($r = Q_{CSD}$). The total gravitational energy of the CSD black hole is given by its CSD charge, whereas for the R-N black hole the total energy is given by its mass parameter [10]. For the charge parameter greater than the mass parameter for the R-N geometry, there is no event horizon and the singularity is naked. However, for the CSD geometry, the singularity is always covered by its event horizon.

In passing we remark that Bekenstein as well as we have not considered the self interaction terms in the conformal scalar field equation and in the corresponding energy-momentum tensor. It is of interest to include them and obtain solutions for CSD black hole. Further, it is essential to study the stability of the CSD black hole.

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